THE RELEVANCE OF MATHEMATICAL MODELS USING GAME THEORY IN ECONOMICS

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Abstract: Mathematical economics is the application of mathematical methods to represent theories and analyze problems in economics. By convention, the applied methods refer to those beyond simple geometry, such as differential and integral calculus, difference and differential equations, matrix algebra, mathematical programming, and other computational methods. An advantage claimed for the approach is its allowing formulation of theoretical relationships with rigor, generality, and simplicity.

Introduction: Mathematics allows economists to form meaningful, testable propositions about wide-ranging and complex subjects which could less easily be expressed informally. Further, the language of mathematics allows economists to make specific, positive claims about controversial or contentious subjects that would be impossible without mathematics. Much of economic theory is currently presented in terms of mathematical economic models, a set of stylized and simplified mathematical relationships asserted to clarify assumptions and implications.

Broad applications include:

- optimization problems as to goal equilibrium, whether of a household, business firm, or policy maker
- static (or equilibrium) analysis in which the economic unit (such as a household) or economic system (such as a market or the economy) is modeled as not changing
- comparative statics as to a change from one equilibrium to another induced by a change in one or more factors
- Dynamic analysis, tracing changes in an economic system over time, for example from economic growth.

Formal economic modeling began in the 19th century with the use of differential calculus to represent and explain economic behavior, such as utility maximization, an early economic application of mathematical optimization. Economics became more mathematical as a discipline throughout the first half of the 20th century, but introduction of new and generalized techniques in the period around the Second World War, as in game theory, would greatly broaden the use of mathematical formulations in economics.

Game theory is the study of strategic behavior of decisions-makers called players in situations where one players decisions may affect the other players. The basic assumption of Game Theory as in economic theory, in general is that decision-makers are rational players, so while pursuing well defined objectives they take into account other decision-makers' s rationality.

Game theory consists of a modeling part and a solution part. Mathematical models of conflicts and of cooperation provide strategic behavioral patterns, and the resulting payoffs to the players are determined according to certain solution concepts.
A game theory may be defined as the collection of rules, regulations and conventions known to planners and observed by the players. A game is then a situation is which two or more participants, the players, confront one another in pursuit of certain conflicting objectives. Some of the players may win and receive a positive pay-off, where as others may lose and get a negative payoff.

A strategy is a plan of action undertaken in the light of own belief about the reaction of the rival. Each player takes into consideration all possible alternative moves in selecting strategy. The final outcome of the game depends jointly on the strategies chosen by the players in the game.

Classification and Description of the Games: There are basically two major categories of games; games of chances and games of strategy. No skill is involved in former type of games. In games of strategy , on the other hand, the outcome depends on the deliberate strategy undertaken by each player. It is here the selection of the strategy skill is called for.

There are several ways of classifying the games of strategy: by the number of players, the number of strategies,, the nature of the payoff function and the nature of preplay negotiations.

a. Depending on the number of players, we may have two –person game, three person-games.........n-person games. Person's as pointed out earlier, does not mean a single individual. It refers to "Participating Party" in the game.
b. another way of classifying games is by the number of strategies, as finite games or infinite games. Infinite games are those in which there are infinite number of strategies available for one or more players..
c. Depending of the pay-off situation, a game can be classified as either constant sum-or non- constant sum. In the former type, the pay-off of all players at the end of the game will always add up to a fixed constant, whatever the strategies chosen by the players. In contrast, the game would become a non- constant one if the payoff to the players in the end do not sum to same constant every time. In a constant-sum game., the constant can be any number, in the special case where it is zero, the game becomes a zero-sum game.
d. There are two main branches of Game Theory. The first is Non-Cooperative Game Theory and the second is Co-Operative Game Theory. In non-Cooperative Games, players see only their own strategic objectives and thus binding agreements among the players are not possible. Therefore strategic interactions among the players ar taken into account. Cooperative game actually is based mainly on agreements to allocate cooperative gains. It ignores the strategic stages leading to coalition building and focuses on the possible results of the cooperation. Such games are relevant when equitable and fair sharing of gains is aimed at.

Non-Cooperative game theory can be applied in situations o duopoly, oligopoly, election contests and the like while the theory of cooperative games is applicable in decision-making in regard to natural resources, environment, water resources, public health and public education.

Payoff Matrix: The two-person constant-sum game can be completely described by means of a payoff matrix derived from a hypothetical example given in the form of following table.

Strategies of Player I	**Strategies of Player II**		
	I	**II**	**III**
I	7	8	4
II	4	7	2

In this, we assume that player I has only two strategies available, but player 2 has three. The total number of possible outcomes is , therefore, player I will receive if both players adopt their respective first strategies. To facilitate reference, let us consider that entry as the a_{11} element of the Payoff matrix A= a_{ij}. Thus the payoff matrix A is defined as;

$$\mathbf{A=} \begin{pmatrix} 7 & 8 & 4 \\ 4 & 7 & 2 \end{pmatrix}$$

Here, the first subscript of a_{ij} always refers to the strategy adopted by player I and the second to the strategy of player II. This convention requires the listing of strategies in the particular form illustrated in the above table i.e those of players in a vertical sequence, and those of player II horizontally. As such a_{23} would mean Payoff to player I when player I adopts strategy two while player II strategy three. From the payoff matrix a_{23}=2.

The payoff matrix for player II can also be derived easily. Since in a constant sum game, the payoff to player II under any given outcome of the game in necessarily equal to that specific constant minus the payoff to player I; matrix b is readily determined from matrix A. Assuming that the constant sum in the present case is 10, then the matrix B can be defined as

$$B= \begin{matrix} (10-7) & (10-8) & (10-4) \\ (10-4) & (10-7) & (10-2) \end{matrix} \quad \begin{matrix} 3 & 2 & 6 \\ 6 & 3 & 8 \end{matrix}$$

In zero – sum game, one player's payoff must be the negative of the pyoff to the other, because a player can only win what his opponent loses. As an economic example of this, consider two firm in a duopolistic market that are striving to lure away each other's established customers. Since such a contest involves exisiting customers only, the number of customers won over by one form must be identical to the number of customers lost by the other. Assume that there are 10 customers in all and that original distribution of customers between two firms is, say

5 to each. Matrices A and B are in terms of number of customers. We can now obtain two payoff matrices A and B in the present zero-sum game by subtracting the original distribution(5) from every element.

$$A = \begin{pmatrix} 2 & 3 & -1 \\ -1 & 2 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & -3 & 1 \\ 1 & -2 & 3 \end{pmatrix}$$

Since we now have

$$a_{ij}+b_{ij} = 0$$

A and B will add up to a 2 X3 zero matrix. In this way we have transformed the constant-sum game into zero-sum game.

Summarizing, constant-sum games are characterized by

$aij+bij = C =$ constant

While zero-sum games are depicted by

$a_{ij}+bij = 0$

Saddle Point: While the structure of a two-person, constant sum game is completely summarized in a single payoff matrix A, this matrix alone does not enable us to tell the final outcome of the game.We make two important assumptions for obtaining the solutions.

1. All players involved in the game theory minimize their risks. In other words, they avoid riskier course of action. Each player seeks to guarantee himself the maximum possible payoff regardless of what the opponent does.
2. Each player has knowledge of the payoff matrix A, but he is ignorant about the exact strategy the opponent plans to adopt.
3. With these assumptions in mind, a conservative player would always proceed thus;
   a. Determine the least payoff he can receive under each of his strategies(i.e the minimum in each row of A)and
   b. Choose the strategy that has the largest minimum.
   c. In this way, he is sure that whatever the opponent does, he will not end up with the worst of all worlds, because he specially Avoids some less favourable outcomes by following the above course of action.

Optimal Strategy for Player I; Applying the above procedure to the payoff matrix A player I will find the minima of the two rows to be 4 and 2 respectively, the maximum among these is 4 which occurs in the first row. This is called the maximin. The optimal strategy for player I, therefore, is his strategy 1.

	Row minimum	maximum
$A = \begin{pmatrix} 7 & 8 & 4 \\ 4 & 7 & 2 \end{pmatrix}$	4 2	4

Optimal Strategy for playerII; Player II shall also seek his own maximin from his own payoff matrix B. If we assume the present game to be 10 sum, then the pypff matrix is B and the maximum is 6, as explained under

$$B=\begin{pmatrix} 3 & 2 & 6 \\ 6 & 3 & 8 \end{pmatrix}$$

Col minimum 3 2 6

Maximin 6

However, in viw of the constant sum nature of the game the maximum among the column miima in matrix B mustyield the same strategy that gives the minimum among the column maxima inmatrix A. We can, therefore, work with matix A instead. In this case, player II will seek to regardless of the strategy chosen by the opponent. Yhus, player II will discard all entries in the payoff matrix A except for the maximum payoff in each column and then select as his optimal strategy the column with the smallest maximum payoff. This is shown as under

$$A=\begin{pmatrix} 7 & 8 & 4 \\ 4 & 7 & 2 \end{pmatrix}$$

Col Maximum 7 8 4

Minimax 4

In sum, therefore, the procedure postulated above essentially amounts to the search for a maximin or a minimax.

In the above example, we observe that the optimal strategy for player I is first one where he obtains a payoff of a least a_{12} =4, whereas player II will pick strategy 3 so that player I gets a payoff of at most a13=4. Thus the two players have agreed upon a solution to the game. This result is arrived at because matrix A satisfies the equation

$$\max_i \min_j (a_{ij}) = \min_j \max_i (a_{ij})$$

These symbols can be explained through the following table considered originally

	J=1	J=2	J=3
i=1	7	8	4
i=2	4	7	2

Now, for i=1, min (a_{ij})=4 and
for i=2, min (a_{2j})=2
Combining both results: min(aij) = 4 which is just a listing of row minima. When the 2 above expression is precede by the expression max it only means to maximize row-wise among the I's.
In other words

$$\max_i \min_j (aij) = \max_i \begin{matrix} 4 \\ 2 \end{matrix}$$

As such the left hand expression of the above equation uis first a concise description of
The procedure for determining the maximin described under "optimal strategy for player I. Similarly the right-hand side of the above equation is the mathematical expression of the minmax procedure outlined under "optimal strategy for player II".

When an element of a payoff matrix serves both as a maximin and as a minimax, it is called a saddle point.in case a saddle point is present in a payoff matrix, it represents the solution of the game and such game is called a strictly determined game.
Mathematically speaking

$$\text{Max } \min(a_{ij}) = \min \ \max(a_{ij}) = a_{ij}$$

Where aij is the saddle point and solution of the game.

Let us take an example of zero-sum game. Here also we assume that player I has a choice of two strategies and player II has a choice of 3 strategies. Player I figures that if he chooses row 1 then the opponent might choose column 2 , resulting in a payoff of 1. Similarly, if he chooses row 2, them he figures that the opponent might choose column I resulting in a payoff -1..these are the row minima as shown below.

$$\begin{array}{lcccc} & & & \text{Row minima} & \text{Maxi min} \\ A= & \begin{pmatrix} 2 & 1 & 4 \\ -1 & 0 & 6 \end{pmatrix} & & \begin{array}{c} 1 \\ -1 \end{array} & 1 \\ \text{Col. Maxima} & 2 \quad 1 \quad 6 & & & \\ \text{Minmax} & 1 & & & \end{array}$$

Maximising over these row minima, player I elects his first strategy guaranteeing a payoff of I or more. Similarly, player II assumes the worst, figures that the opponent might select the first row if he chooses column 1 or 2 and the second row if he chooses column 3, leading to the column 2,1 and 6 as shown above. Minimising over these column maxima, player II chooses his second strategy, guaranteeing a loss of not more than 1. Thus, in this game the choices are consistent.

$$\text{Max } \min \ a_{ij} = \min \ \max \ a_{ij} = 1$$

And tha saddle point entry I, is the value of game (to player I)

Mixed Strategy; The Case of No Saddle Point

Not all two-person zero-sum(or constant-sum) game are strictly determined. Let us now explore the case where the paypff matrix contains no saddle point. Consider a game with the following payoff matrix:

$$\begin{array}{lcc} & & \text{Row minima} \\ A= & \begin{pmatrix} 6 & -2 & 3 \\ -4 & 5 & 4 \end{pmatrix} & 2 \\ \text{Col. Maxima} & & 6 \quad 5 \quad 4 \end{array}$$

In this case, max. min aij = -2<4 = min aij. If the players follow the rules developed thus far, player 11 selescts strategy 1 and expects player II to select strategy 2 and a payoff -2, while player II selects strategy 3 and expects player I to select strategy two and a payoff of 4. The outcome is aij=3, which neither player expected.

Furthermore, if player II does select his third strategy, then player I would do better selecting his second, not his first strategy. Similarly if player I does select his first strategy, then player II would do better selecting his seconds, not his third strategy. The solution concept as outlined so far seems to fail in such games.

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